Extensions of Barrier Sets to Nonzero Roots of the Matching Polynomials

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Abstract

In matching theory, barrier sets (also known as Tutte sets) have been studied extensively due to its connection to maximum matchings in a graph. In this paper, we first define θ-barrier sets. Our definition of a θ-barrier set is slightly different from that of a barrier set. However we show that θ-barrier sets and barrier sets have similar properties. In particular, we prove a generalized Berge's Formula and give a characterization for the set of all θ-special vertices in a graph.

KEYWORDS: matching polynomial, Gallai-Edmonds Decomposition, barrier sets, extreme sets

1 Introduction

All the graphs in this paper are simple and finite.

Definition 1.1. An r-*matching* in a graph G is a set of r edges, no two of which have a vertex in common. The number of r-matchings in G will be denoted by $p(G, r)$. We set $p(G, 0) = 1$ and define the *matching polynomial* of G by

$$\mu(G, x) = \sum_{r=0}^{\lfloor n/2 \rfloor} (-1)^r p(G, r) x^{n-2r}.$$

We shall denote the multiplicity of θ as a root of $\mu(G, x)$ by $\mathrm{mult}(\theta, G)$. Let $u \in V(G)$, the graph obtained from G by deleting the vertex u and all edges that contain u will be denoted by $G \setminus u$. Inductively if $u_1, \ldots, u_k \in V(G)$, $G \setminus u_1 \ldots u_k = (G \setminus u_1 \ldots u_{k-1}) \setminus u_k$. Note that the order of which vertex is being deleted first is not important, that is, if $i_1, \ldots, i_k$ is a permutation of $1, \ldots, k$, we have $G \setminus u_1 \ldots u_k = G \setminus u_{1_1} \ldots u_{i_k}$. Furthermore if $X = \{u_1, \ldots, u_k\}$, $G \setminus X = G \setminus u_1 \ldots u_k$.

The followings are properties of $\mu(G, x)$.

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Theorem 1.2. (Theorem 1.1 on p. 2 of [1])

(a) $\mu(G \cup H, x) = \mu(G, x)\mu(H, x)$ *where* G *and* H *are disjoint graphs,*

(b) $\mu(G, x) = \mu(G \setminus e, x) - \mu(G \setminus uv, x)$ *if* $e = \{u, v\}$ *is an edge of* G*,*

(c) $\mu(G, x) = x\mu(G \setminus u, x) - \sum_{i \sim u} \mu(G \setminus ui, x)$ *where* $i \sim u$ *means* i *is adjacent to* u*,*

(d) $\dfrac{d}{dx}\mu(G, x) = \displaystyle\sum_{i \in V(G)} \mu(G \setminus i, x)$ *where* $V(G)$ *is the vertex set of* G*.*

It is well known that all roots of $\mu(G, x)$ are real. Throughout, let θ be a real number and $\mathrm{mult}(\theta, G)$ denote the multiplicity of θ as a root of $\mu(G, x)$. In particular, $\mathrm{mult}(\theta, G) = 0$ if and only if θ is not a root of $\mu(G, x)$. By Theorem 5.3 on p. 29 and Theorem 1.1 on p. 96 of [1], one can easily deduce the following lemma.

Lemma 1.3. *Let* G *be a graph and* $u \in V(G)$*. Then*

$$\mathrm{mult}(\theta, G) - 1 \leq \mathrm{mult}(\theta, G \setminus u) \leq \mathrm{mult}(\theta, G) + 1.$$

As a consequence of Lemma 1.3, we can classify the vertices in a graph with respect to θ as follows:

Definition 1.4. (see [2, Section 3]) For any $u \in V(G)$,

(a) u is θ-*essential* if $\mathrm{mult}(\theta, G \setminus u) = \mathrm{mult}(\theta, G) - 1$,

(b) u is θ-*neutral* if $\mathrm{mult}(\theta, G \setminus u) = \mathrm{mult}(\theta, G)$,

(c) u is θ-*positive* if $\mathrm{mult}(\theta, G \setminus u) = \mathrm{mult}(\theta, G) + 1$.

Furthermore if u is not θ-essential but it is adjacent to some θ-essential vertex, we say u is θ-special.

It turns out that θ-special vertices play an important role in the Gallai-Edmonds Decomposition of a graph (see [3]). One of our main result is a characterization of the set of these vertices in terms of θ-barriers.

Note that if $\mathrm{mult}(\theta, G) = 0$ then for any $u \in V(G)$, u is either θ-neutral or θ-positive and no vertices in G can be θ-special. By Corollary 4.3 of [2], a θ-special vertex is θ-positive. Therefore

$$V(G) = D_\theta(G) \cup A_\theta(G) \cup P_\theta(G) \cup N_\theta(G),$$

where

$D_\theta(G)$ is the set of all θ-essential vertices in G,

$A_\theta(G)$ is the set of all θ-special vertices in G,

$N_\theta(G)$ is the set of all θ-neutral vertices in G,

$P_\theta(G) = Q_\theta(G) \setminus A_\theta(G)$, where $Q_\theta(G)$ is the set of all θ-positive vertices in G,

is a partition of $V(G)$. Note that there is no 0-neutral vertices. So $N_0(G) = \varnothing$ and $V(G) = D_0(G) \cup A_0(G) \cup P_0(G)$.

Definition 1.5. (see [2, Section 3]) A graph G is said to be θ-critical if all vertices in G are θ-essential and $\mathrm{mult}(\theta, G) = 1$.

The Gallai-Edmonds Structure Theorem describes a certain canonical decomposition of $V(G)$ with respect to the zero root of $\mu(G,x)$. In [3], Chen and Ku proved the Gallai-Edmonds Structure Theorem for graph with any root θ.

Theorem 1.6. (Theorem 1.5 of [3]) *Let G be a graph with θ a root of $\mu(G,x)$. If $u \in A_\theta(G)$ then*

(i) $D_\theta(G \setminus u) = D_\theta(G)$,

(ii) $P_\theta(G \setminus u) = P_\theta(G)$,

(iii) $N_\theta(G \setminus u) = N_\theta(G)$,

(iv) $A_\theta(G \setminus u) = A_\theta(G) \setminus \{u\}$.

Theorem 1.7. (Theorem 1.7 of [3]) *If G is connected and every vertex of G is θ-essential then* $\mathrm{mult}(\theta, G) = 1$.

By Theorem 1.6 and Theorem 1.7, it is not hard to deduce the following whose proof is omitted. For convenience, a connected component will be called a component.

Corollary 1.8.

(i) $A_\theta(G \setminus A_\theta(G)) = \varnothing$, $D_\theta(G \setminus A_\theta(G)) = D_\theta(G)$, $P_\theta(G \setminus A_\theta(G)) = P_\theta(G)$, *and* $N_\theta(G \setminus A_\theta(G)) = N_\theta(G)$.

(ii) $G \setminus A_\theta(G)$ *has exactly* $|A_\theta(G)| + \mathrm{mult}(\theta, G)$ *θ-critical components.*

(iii) If H is a component of $G \setminus A_\theta(G)$ then either H is θ-critical or $\mathrm{mult}(\theta, H) = 0$.

(iv) The subgraph induced by $D_\theta(G)$ consists of all the θ-critical components in $G \setminus A_\theta(G)$.

Let G be a graph. The number of odd components in G is denoted by $c_{odd}(G)$. Recall the following famous Berge's Formula.

Theorem 1.9. $\mathrm{mult}(0, G) = \max_{X \subseteq V(G)} c_{odd}(G \setminus X) - |X|$.

Definition 1.10. Motivated by the Berge's Formula, a *barrier set* is defined to be a set $X \subseteq V(G)$ for which $\mathrm{mult}(0, G) = c_{odd}(G \setminus X) - |X|$. An *extreme* set is defined to be the set for which $\mathrm{mult}(0, G \setminus X) = \mathrm{mult}(0, G) + |X|$.

Properties of extreme and barrier sets can be found in [4, Section 3.3]. In fact a barrier set is an extreme set. An extreme set is not necessary a barrier set, but it can be shown that an extreme set is contained in some barrier set. In general the union or intersection of two barrier sets is not a barrier set. However it can be shown that the intersection of two (inclusionwise) maximal barriers set is a barrier set. $A_0(G)$ is a barrier and extreme set. It can be shown that $A_0(G)$ is in fact the intersection of all the maximal barrier sets in G. Here we extend this fact to $A_\theta(G)$:

Theorem 1.11. *Suppose $N_\theta(G) = \varnothing$. Then $A_\theta(G)$ is the intersection of all maximal θ-barrier sets in G.*

2 Properties of θ-barrier sets

The number of θ-critical components in G is denoted by $c_\theta(G)$. An immediate consequence of part (a) of Theorem 1.2 and Theorem 1.7 is the following inequality which is used frequently.

$$\text{mult}(\theta, G \setminus X) \geq c_\theta(G \setminus X) \quad \text{for any} \quad X \subseteq V(G). \tag{1}$$

We prove the following analogue of Berge's Formula.

Theorem 2.1. [Generalized Berge's Formula]

$$\text{mult}(\theta, G) = \max_{X \subseteq V(G)} c_\theta(G \setminus X) - |X|.$$

Proof. We claim that, $c_\theta(G \setminus X) \leq |X| + \text{mult}(\theta, G)$ for all $X \subseteq V(G)$. Suppose the contrary. Then $c_\theta(G \setminus X) > |X| + \text{mult}(\theta, G)$ for some $X \subseteq V(G)$. Recall that $\text{mult}(\theta, G \setminus X) \geq c_\theta(G \setminus X)$. Together with Lemma 1.3, we have $\text{mult}(\theta, G) \geq \text{mult}(\theta, G \setminus X) - |X| > \text{mult}(\theta, G)$, a contradiction. Hence $c_\theta(G \setminus X) \leq |X| + \text{mult}(\theta, G)$ for all $X \subseteq V(G)$.

Now it is sufficient to show that there is a set $X \subseteq V(G)$ for which $\text{mult}(\theta, G) = c_\theta(G \setminus X) - |X|$. By (ii) of Corollary 1.8 and taking $X = A_\theta(G)$, we are done. □

Definition 2.2. Motivated by the Generalized Berge's Formula, we define a *θ-barrier set* to be a set $X \subseteq V(G)$ for which $\text{mult}(\theta, G) = c_\theta(G \setminus X) - |X|$.

We define a *θ-extreme set* to be a set $X \subseteq V(G)$ for which $\text{mult}(\theta, G \setminus X) = \text{mult}(\theta, G) + |X|$.

Note that the definitions of 0-extreme set and extreme set coincide. But the definitions of 0-barrier set and barrier set are different. Our next proposition shows that a 0-barrier set is a barrier set.

Proposition 2.3. *A 0-barrier set is a barrier set.*

Proof. Let X be a 0-barrier set. Then $c_0(G \setminus X) = \text{mult}(0, G) + |X|$. Note that $c_0(G \setminus X) \leq c_{odd}(G \setminus X)$. Using Theorem 1.9, we conclude that $c_{odd}(G \setminus X) = \text{mult}(0, G) + |X|$. Hence X is a barrier set. □

The converse of Proposition 2.3 is not true. In Figure 1, $X = \{u, v\}$ is a barrier set in G but it is not a 0-barrier set.

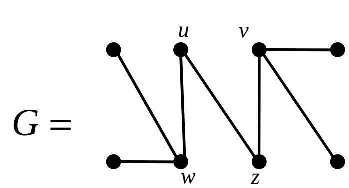

Figure 1.

However we have a weak converse of Proposition 2.3.

Proposition 2.4. *A (inclusionwise) maximal barrier set is a maximal* 0*-barrier set.*

Proof. Let X be a maximal barrier set. Note that $|X| + \text{mult}(0, G) \geq \text{mult}(0, G \setminus X) \geq c_{odd}(G \setminus X) = |X| + \text{mult}(0, G)$, where the first inequality follows from Lemma 1.3 and the last inequality follows from the fact that X is a barrier set. Therefore, equality holds throughout whence $\text{mult}(0, G \setminus X) = c_{odd}(G \setminus X)$ and 0 is a root of multiplicity 1 in each of the odd components in $G \setminus X$.

We claim that an odd component in $G \setminus X$ is 0-critical. Suppose the contrary. Let H be an odd component in $G \setminus X$ and H is not 0-critical. Then $A_0(H) \neq \varnothing$. Now $\text{mult}(0, H) = 1$. By (ii) of Corollary 1.8, $c_0(H \setminus A_0(H)) = |A_0(H)| + 1$. Since $c_0(H \setminus A_0(H)) \leq c_{odd}(H \setminus A_0(H))$, by Theorem 1.9, we conclude that $c_{odd}(H \setminus A_0(H)) = |A_0(H)| + 1$. Therefore $c_{odd}(G \setminus (X \cup A_0(H)) = c_{odd}(G \setminus X) - 1 + c_{odd}(H \setminus A_0(H)) = |X| + \text{mult}(0, G) - 1 + |A_0(H)| + 1 = |X \cup A_0(H)| + \text{mult}(0, G)$. But then $X \cup A_0(H)$ is a barrier set in G, a contrary to the maximality of X. Hence an odd component in $G \setminus X$ must be 0-critical. This means that $c_{odd}(G \setminus X) = c_0(G \setminus X)$ and X is a 0-barrier set. By Proposition 2.3, we conclude that X must be a maximal 0-barrier set. ☐

Now we shall study the properties of θ-barrier and θ-extreme sets.

Lemma 2.5. *A subset of a θ-extreme set is a θ-extreme set.*

Proof. Let X be an θ-extreme set and $Y \subseteq X$. Now $\text{mult}(\theta, G \setminus X) = \text{mult}(\theta, G) + |X|$. By Lemma 1.3, $\text{mult}(\theta, G \setminus Y) \leq \text{mult}(\theta, G) + |Y|$. If Y is not θ-extreme then $\text{mult}(\theta, G \setminus Y) < \text{mult}(\theta, G) + |Y|$, and by Lemma 1.3 again, $\text{mult}(\theta, G \setminus X) \leq \text{mult}(\theta, G \setminus Y) + |X \setminus Y| < \text{mult}(\theta, G) + |X|$, a contradiction. Hence a subset of an θ-extreme set is θ-extreme. ☐

Lemma 2.6. *If X is a θ-barrier [θ-extreme] set and $Y \subseteq X$ then $X \setminus Y$ is a θ-barrier [θ-extreme] set in $G \setminus Y$.*

Proof. Note that $c_\theta(G \setminus X) = |X| + \text{mult}(\theta, G)$. By Theorem 2.1 and Lemma 1.3, $c_\theta(G \setminus X) \leq |X \setminus Y| + \text{mult}(\theta, G \setminus Y) \leq |X \setminus Y| + \text{mult}(\theta, G) + |Y| = |X| + \text{mult}(\theta, G)$. Hence $c_\theta(G \setminus X) = |X \setminus Y| + \text{mult}(\theta, G \setminus Y)$ and $X \setminus Y$ is a θ-barrier set in $G \setminus Y$. ☐

Lemma 2.7. *Every θ-extreme set of G lies in a θ-barrier set.*

Proof. Let X be a θ-extreme set and $T = A_\theta(G \setminus X) \cup X$. Then

$$\begin{aligned}
c_\theta(G \setminus T) &= c_\theta(G \setminus (A_\theta(G \setminus X) \cup X)) \\
&= c_\theta((G \setminus X) \setminus A_\theta(G \setminus X)) \\
&= |A_\theta(G \setminus X)| + \text{mult}(\theta, G \setminus X) \text{ (by (ii) of Corollary 1.8)} \\
&= |A_\theta(G \setminus X)| + \text{mult}(\theta, G) + |X| \text{ (X is θ-extreme)} \\
&= |T| + \text{mult}(\theta, G),
\end{aligned}$$

and hence T is a θ-barrier set. ☐

Lemma 2.8. *Let X be a θ-barrier set. Then X is a θ-extreme set.*

Proof. Recall from (1) that $\operatorname{mult}(\theta, G\setminus X) \geq c_\theta(G\setminus X)$. Since $c_\theta(G\setminus X) = |X| + \operatorname{mult}(\theta, G)$, by Lemma 1.3, we have

$$\operatorname{mult}(\theta, G) \geq \operatorname{mult}(\theta, G \setminus X) - |X| \geq c_\theta(G \setminus X) - |X| = \operatorname{mult}(\theta, G).$$

Hence $\operatorname{mult}(\theta, G \setminus X) = \operatorname{mult}(\theta, G) + |X|$ and so X is a θ-extreme set. □

Note that in general a θ-extreme set is not a θ-barrier set. In Figure 1, $X_1 = \{u\}$ is a 0-extreme set but it is not a 0-barrier set.

Lemma 2.9. *Let X be a θ-barrier set and H be a component of $G \setminus X$. Then either H is θ-critical or* $\operatorname{mult}(\theta, H) = 0$.

Proof. Note that $c_\theta(G \setminus X) = |X| + \operatorname{mult}(\theta, G)$. By Lemma 2.8, X is a θ-extreme set. Therefore $\operatorname{mult}(\theta, G\setminus X) = \operatorname{mult}(\theta, G) + |X| = c_\theta(G\setminus X)$. Now if H is not θ-critical and $\operatorname{mult}(\theta, H) > 0$, then by (1), $\operatorname{mult}(\theta, G \setminus X) > c_\theta(G \setminus X)$, a contradiction. Hence either H is θ-critical or $\operatorname{mult}(\theta, H) = 0$. □

Lemma 2.10. *Let X be a maximal θ-barrier set. Let H be a component of $G\setminus X$ and* $\operatorname{mult}(\theta, H) = 0$. *Then for all $u \in V(H)$, u is θ-neutral in H. Furthermore for all $Y \subseteq V(H)$ and $Y \neq \varnothing$, $c_\theta(H \setminus Y) \leq |Y| - 1$.*

Proof. Suppose H has a θ-positive vertex, say u. Then $\operatorname{mult}(\theta, H \setminus u) = 1$. By (ii) of Corollary 1.8, $c_\theta((H \setminus u) \setminus A_\theta(H \setminus u)) = |A_\theta(H \setminus u)| + \operatorname{mult}(\theta, H \setminus u) = |A_\theta(H \setminus u)| + 1$. But then

$$\begin{aligned} c_\theta(G \setminus (X \cup \{u\} \cup A_\theta(H \setminus u)) &= c_\theta(G \setminus X) + c_\theta((H \setminus u) \setminus A_\theta(H \setminus u)) \\ &= |X| + \operatorname{mult}(\theta, G) + |A_\theta(H \setminus u)| + 1 \\ &= |X \cup \{u\} \cup A_\theta(H \setminus u)| + \operatorname{mult}(\theta, G), \end{aligned}$$

and so $X \cup \{u\} \cup A_\theta(H \setminus u)$ is a θ-barrier in G, a contrary to the maximality of X. Hence for all $u \in V(H)$, u is θ-neutral in H.

Since $Y \neq \varnothing$, there is a $y \in Y$. Let $Y' = Y \setminus y$ and $H' = H \setminus y$. Note that $\operatorname{mult}(\theta, H \setminus y) = 0$ since y is θ-neutral in H. By Theorem 2.1, $c_\theta(H' \setminus Y') \leq |Y'|$. Since $H \setminus Y = H' \setminus Y'$, we have $c_\theta(H \setminus Y) \leq |Y| - 1$. □

Lemma 2.11. *Let G be θ-critical. Then for all $Y \subseteq V(G)$ and $Y \neq \varnothing$, $c_\theta(G \setminus Y) \leq |Y| - 1$.*

Proof. Since $Y \neq \varnothing$, there is a $y \in Y$. Let $Y' = Y \setminus y$ and $G' = G \setminus y$. Note that $\operatorname{mult}(\theta, G \setminus y) = 0$ since y is θ-essential in G. By Theorem 2.1, $c_\theta(G' \setminus Y') \leq |Y'|$. Since $G \setminus Y = G' \setminus Y'$, we have $c_\theta(G \setminus Y) \leq |Y| - 1$. □

In general the union or intersection of two θ-barrier sets is not necessary a θ-barrier set. In Figure 1, $X_2 = \{u, v, w\}$ and $X_3 = \{v, w, z\}$ are two 0-barrier sets. But $X_2 \cap X_3$ and $X_2 \cup X_3$ are not a 0-barrier set. However the intersection of two maximal θ-barrier sets is a θ-barrier set.

Theorem 2.12. *The intersection of two maximal θ-barrier sets is a θ-barrier set.*

Proof. Let X and Y be two maximal θ-barrier sets. Let $G_1, G_2, \ldots, G_k$ be all the θ-critical components of $G \setminus X$ and $H_1, H_2, \ldots, H_m$ be all the components of $G \setminus Y$. Note that $k = |X| + \mathrm{mult}(\theta, G)$. Let $X_i = X \cap V(H_i)$, $Y_i = Y \cap V(G_i)$ and $Z = X \cap Y$. By relabelling if necessary we may assume that $X_1, \ldots, X_{m_1} \neq \varnothing$, $Y_1, \ldots, Y_{k_1} \neq \varnothing$, but $X_{m_1+1} = \cdots = X_m = Y_{k_1+1} = \cdots = Y_k = \varnothing$, and also that $k_1 \leq m_1$. Note that $G_{k_1+1}, \ldots, G_k$ are θ-critical components in $(G \setminus X) \setminus Y$. So each of them is contained in a component of $G \setminus Y$. Now let us count the number of G_i's where $k_1 + 1 \leq i \leq k$ that are contained in some H_j.

Suppose $m_1 + 1 \leq j \leq m$. Then H_j is a component in $(G \setminus X) \setminus Y$. So if $G_i \subseteq H_j$, we must have $G_i = H_j$. Furthermore G_i is a component of $G \setminus Z$. By Theorem 2.1, the number of such G_i's is at most $c_\theta(G \setminus Z) \leq |Z| + \mathrm{mult}(\theta, G)$.

Suppose $1 \leq j \leq m_1$. Let $G_{i_1}, \ldots, G_{i_t}$ be all the G_i's that are contained in H_j. Then $G_{i_1}, \ldots, G_{i_t}$ are θ-critical components in $H_j \setminus X_j$. By Lemma 2.9, H_j is either θ-critical or $\mathrm{mult}(\theta, H) = 0$. If $\mathrm{mult}(\theta, H) = 0$, we have, by Lemma 2.10, $c_\theta(H_j \setminus X_j) \leq |X_j| - 1$. If H_i is θ-critical, we have, by Lemma 2.11, $c_\theta(H_j \setminus X_j) \leq |X_j| - 1$. Therefore in either cases, we have $t \leq |X_j| - 1$.

The number of G_i's where $k_1 + 1 \leq i \leq k$ that are disjoint from Y is at most

$$\begin{aligned} c_\theta(G \setminus Z) + \sum_{j=1}^{m_1} (|X_j| - 1) &\leq |Z| + \mathrm{mult}(\theta, G) + |X \setminus Z| - m_1 \\ &= |X| + \mathrm{mult}(\theta, G) - m_1 \\ &= k - m_1 \\ &\leq k - k_1. \end{aligned}$$

Since this number is exactly $k - k_1$, we infer that equality must hold throughout. Hence $c_\theta(G \setminus Z) = |Z| + \mathrm{mult}(\theta, G)$ and Z is a θ-barrier set. ☐

3 Characterizations of $A_\theta(G)$

A characterization of $A_\theta(G)$ is that it is the minimal (inclusionwise) θ-barrier set (see Theorem 3.5). Furthermore if $N_\theta(G) = \varnothing$, we have another characterization of $A_\theta(G)$, that is, it is the intersection of all maximal θ-barrier sets in G (see Theorem 3.6).

Lemma 3.1. *If X is a θ-barrier or a θ-extreme set then $X \subseteq A_\theta(G) \cup P_\theta(G)$.*

Proof. By Lemma 2.8, we may assume X is a θ-extreme. Let $x \in X$. By Lemma 2.5, $\{x\}$ is a θ-extreme set. Therefore $\mathrm{mult}(\theta, G \setminus x) = \mathrm{mult}(\theta, G) + 1$ and x is θ-positive. So $x \in A_\theta(G) \cup P_\theta(G)$ and $X \subseteq A_\theta(G) \cup P_\theta(G)$. ☐

Lemma 3.2. *Let X be a θ-barrier set. If $X \subseteq A_\theta(G)$ then $X = A_\theta(G)$.*

Proof. Note that $c_\theta(G \setminus X) = \mathrm{mult}(\theta, G) + |X|$. By Lemma 2.9, we conclude that $A_\theta(G \setminus X) = \varnothing$. By Theorem 1.6, $A_\theta(G \setminus X) = A_\theta(G) \setminus X$. Hence $X = A_\theta(G)$. ☐

We shall require the following result of Godsil [2].

Theorem 3.3. (Theorem 4.2 of [2]) *Let θ be a root of $\mu(G,x)$ with non-zero multiplicity k and let u be a θ-positive vertex in G. Then*

(a) if v is θ-essential in G then it is θ-essential in $G \setminus u$;

(b) if v is θ-positive in G then it is θ-essential or θ-positive in $G \setminus u$;

(c) if u is θ-neutral in G then it is θ-essential or θ-neutral in $G \setminus u$.

Lemma 3.4. *Let $u \in P_\theta(G)$. Then $A_\theta(G) \subseteq A_\theta(G \setminus u)$.*

Proof. If $A_\theta(G) = \varnothing$, then we are done. Suppose $A_\theta(G) \neq \varnothing$. Let $v \in A_\theta(G)$. Then v is adjacent to a θ-essential vertex w. By Theorem 3.3, w is θ-essential in $G \setminus u$ and v is either θ-positive or θ-essential in $G \setminus u$. Suppose v is θ-essential in $G \setminus u$. Then $\mathrm{mult}(\theta, G \setminus uv) = \mathrm{mult}(\theta, G)$. By Theorem 1.6, $u \in P_\theta(G) = P_\theta(G \setminus v)$. Since v is θ-special in G, v is θ-positive in G (see Corollary 4.3 of [2]). So $\mathrm{mult}(\theta, G \setminus uv) = \mathrm{mult}(\theta, G) + 2$, a contradiction. Therefore v is θ-positive in $G \setminus u$. Since v is adjacent to w, we must have $v \in A_\theta(G \setminus u)$. Hence $A_\theta(G) \subseteq A_\theta(G \setminus u)$. $\square$

Theorem 3.5. *Let X be a θ-barrier set in G. Then $A_\theta(G) \subseteq X$. In particular, $A_\theta(G)$ is the minimal θ-barrier set.*

Proof. By Lemma 3.1, $X \subseteq A_\theta(G) \cup P_\theta(G)$. We shall prove the result by induction on $|X \cap P_\theta(G)|$. Suppose $|X \cap P_\theta(G)| = 0$. Then $X \subseteq A_\theta(G)$ and by Lemma 3.2, $X = A_\theta(G)$. Suppose $|X \cap P_\theta(G)| \geq 1$. We may assume that if X' is a θ-barrier set in G' with $|X' \cap P_\theta(G')| < |X \cap P_\theta(G)|$, then $A_\theta(G') \subseteq X'$.

Let $x \in X \cap P_\theta(G)$. By Lemma 2.6, $X' = X \setminus x$ is a θ-barrier set in $G' = G \setminus x$. By Lemma 3.1 and Lemma 3.4, we have $X' \subseteq A_\theta(G') \cup P_\theta(G')$ and $A_\theta(G) \subseteq A_\theta(G')$. Therefore $|X' \cap P_\theta(G')| < |X \cap P_\theta(G)|$. By induction $A_\theta(G') \subseteq X'$. Hence $A_\theta(G) \subseteq X$. $\square$

In general, $A_\theta(G)$ is not the intersection of all maximal θ-barrier sets in G. For instance, in Figure 2, $\mathrm{mult}(\sqrt{3}, G) = 0$ and $A_{\sqrt{3}}(G) = \varnothing$. Now $\{u\}$ is the only maximal $\sqrt{3}$-barrier set. But $A_{\sqrt{3}}(G) \neq \{u\}$. However we can show that $A_\theta(G)$ is the intersection of all maximal θ-barrier sets in G if $N_\theta(G) = \varnothing$.

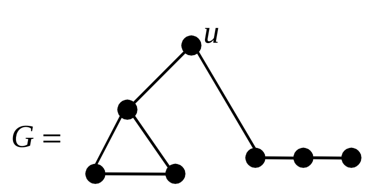

Figure 2.

Theorem 3.6. *Suppose $N_\theta(G) = \varnothing$. Then $A_\theta(G)$ is the intersection of all maximal θ-barrier sets in G.*

Proof. By Theorem 3.5, $A_\theta(G)$ is contained in the intersection of all maximal θ-barriers in G. It is sufficient to show that for each $x \in V(G) \setminus A_\theta(G)$ there is a maximal barrier that does not contain x. If $x \in D_\theta(G)$, by Lemma 3.1, x is not contained in any θ-barriers and thus any maximal θ-barriers. Suppose $x \in P_\theta(G)$. Then x is contained in a component H in $G \setminus A_\theta(G)$ with $\mathrm{mult}(\theta, H) = 0$. Note that $|V(H)| \geq 2$, for $x \in P_\theta(G) = P(G \setminus A_\theta(G))$ and $\mathrm{mult}(\theta, H \setminus x) = 1$ (see Theorem 1.6). By (c) of Theorem 1.2 and the fact that $\mathrm{mult}(\theta, H) = 0$, we deduce that there is a vertex $y \in V(H \setminus x)$ for which $\mathrm{mult}(\theta, H \setminus xy) = 0$. Now $y \in P_\theta(G)$ for $N_\theta(G) = \varnothing$. Furthermore x is θ-essential in $H \setminus y$. Therefore $x \notin A_\theta(H \setminus y)$ and by (ii) of Corollary 1.8, $c_\theta((H \setminus y) \setminus A_\theta(H \setminus y)) = |A_\theta(H \setminus y)| + 1$. Hence

$$\begin{aligned} c_\theta(G \setminus (A_\theta(G) \cup \{y\} \cup A_\theta(H \setminus y))) &= c_\theta(G \setminus A_\theta(G)) + c_\theta((H \setminus y) \setminus A_\theta(H \setminus y)) \\ &= |A_\theta(G)| + \mathrm{mult}(\theta, G) + |A_\theta(H \setminus y)| + 1 \\ &= |A_\theta(G) \cup \{y\} \cup A_\theta(H \setminus y)| + \mathrm{mult}(\theta, G), \end{aligned}$$

and so $A_\theta(G) \cup \{y\} \cup A_\theta(H \setminus y)$ is a θ-barrier set not containing x. Let Z be a maximal θ-barrier set containing $Y = A_\theta(G) \cup \{y\} \cup A_\theta(H \setminus y)$. By Lemma 2.6, $Z \setminus Y$ is a θ-barrier set in $G \setminus Y$. Using Theorem 1.6 and the fact that x is θ-essential in $H \setminus y$, we can deduce that $x \in D_\theta(G \setminus Y)$. By Lemma 3.1, we conclude that $x \notin Z \setminus Y$ and hence $x \notin Z$. The proof of the theorem is completed. $\square$

Since $N_0(G) = \varnothing$, by Theorem 3.6 and Proposition 2.4, we deduce the following classical result.

Corollary 3.7. (Theorem 3.3.15 of [4]) *$A_0(G)$ is the intersection of all maximal barrier sets in G.*

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